Effect of Inelastic Collisions on the Tail of the Electron Velocity Distribution

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The effect of inelastic electron collisions with two-level atoms, having either two bound levels or one bound level and a continuum, on the electron velocity distribution is reconsidered. Attention is paid to the role played by physical parameters like cross sections, atomic occupation numbers, and temperature. Distribution functions and collision rates are given in terms of easily accessible tabulated functions.

1. Introduction

The electron velocity distribution of a stationary plasma deviates from a Maxwellian one as a result of inelastic collisions of electrons with atoms provided that the occupation numbers of the atomic bound and continuum levels deviate from Boltzmann and Saha distributions, respectively, and that inelastic electron collisions are sufficiently frequent compared to elastic ones. This fact is important for plasma spectroscopy because the collisional excitation and ionization rates depend on the electron velocity distribution.

Investigations of this topic are due to BOHM and ALLER [1], KAGAN and LJAGUSTSCHENKO [2], ROTHER [3], WOJACZEK [4], BIBERMAN, VOROBEV, and YAKUBOV [5], and PEYRAUD [6]. The present paper takes up the discussion by reconsidering the effect of inelastic collisions with two-level atoms on the tail of the electron velocity distribution. This is done for three reasons: 1. to display the role of the physical parameters involved, 2. to show that free-bound transitions can be treated in essentially the same way as bound-bound ones, and 3. to study quantitatively the influence that deviations of the atomic occupation numbers from their thermal values have on the effect considered. The results obtained may facilitate the search for realistic situations in which the electron velocity distribution can be expected to differ appreciably from a Maxwellian one.

2. System

Our system is a stationary homogeneous and isotropic plasma containing two-level atoms with either two bound levels or one bound level and a continuum. The only processes considered are elastic electron-electron and inelastic electron-atom collisions. The excitation or ionization energy of the atoms is supposed to be much greater than the mean kinetic energy of the electrons so that only the tail of the electron distribution function is affected by inelastic collisions. Atomic occupation numbers are treated as free parameters.

3. Stationarity Equation

The stationarity of the electron distribution function requires the collision terms due to elastic and inelastic collisions to cancel each other:

$$(\partial f/\partial t)_{\text{el}} + (\partial f/\partial t)_{\text{inel}} = 0\,. \tag{1}$$

3.1. Elastic Collisions

The Fokker-Planck collision term due to elastic electron-electron collisions, valid for the high-energy tail of the isotropic distribution $f(E)$ [7], is in terms of kinetic energy E rather than velocity given by

$$\left(\frac{\partial f}{\partial t}\right)_{\text{el}}^{\text{ee}} = \frac{m^{3/2}\,n_{\text{e}}\,\Gamma}{2^{1/2}}\,\frac{\mathrm{d}}{\mathrm{d}E}\left[\frac{f(E)}{E^{1/2}} + k\,T\,\frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{f(E)}{E^{1/2}}\right)\right] \tag{2}$$

where $\Gamma = \dfrac{4\,\pi\,e_0{}^4}{m^2}\ln\Lambda\,,\quad \Lambda = \dfrac{3\,(k\,T)^{3/2}}{2\,\pi^{1/2}\,e_0{}^3\,n_{\text{e}}{}^{1/2}}$ (3)

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[1] D. BOHM and L. H. ALLER, Astrophys. J. 105, 1 [1947].
[2] J. M. KAGAN and R. J. LJAGUSTSCHENKO, J. Tekh. Fiz. 31, 455 [1961]; 32, 192 [1962]; 34, 821 [1964].
[3] H. ROTHER, Z. Physik 173, 441 [1963].
[4] K. WOJACZEK, Beitr. Plasmaphys. 5, 181 [1965].
[5] L. M. BIBERMAN, V. S. VOROBEV, and I. T. YAKUBOV, High Temperature 6, 359 [1968].
[6] N. PEYRAUD, Phys. Letters 27 A, 410 [1968]; Thèse, Faculté des Sciences de Paris, Paris 1969.
[7] M. N. ROSENBLUTH, W. M. MACDONALD, and D. L. JUDD, Phys. Rev. 107, 1 [1957].

(n_e = electron density, m = electron mass, e_0 = elementary charge). Collision term (2) implies that $f(E)$ differs from a Maxwell distribution only for energies $E \gg kT$ so that it still makes sense to speak of an electron temperature T. Furthermore, it has the property of leaving the Maxwell distribution $f_M(E) \sim E^{1/2} e^{-E/kT}$ unaffected:

$$(\partial f_M/\partial t)_{el}^{ee} = 0 .$$

3.2. *Inelastic Collisions*

3.2.1. Bound-bound transitions. The ground state of the two-level atom will be denoted by "1", the excited state by "2". The collision term due to inelastic electron-atom collisions is for energies $E \geqq$ excitation energy E_{12} given by

$$\left(\frac{\partial f}{\partial t}\right)_{\text{inel}}^{\text{bb}} = -n_1 \left(\frac{2E}{m}\right)^{1/2} Q_{12}(E)\, f(E) + n_2 \left(\frac{2}{m}(E-E_{12})\right)^{1/2} Q_{21}(E-E_{12})\, f(E-E_{12}) \tag{4}$$

(n_1, n_2 = atom densities; Q_{12}, Q_{21} = cross sections for collisional excitation and deexcitation). Two further terms describing the creation and destruction of electrons of energy E by means of excitation collisions of electrons of energy $E + E_{12}$ and de-excitation collisions of electrons of energy E, respectively, have been neglected since they are small for $E_{12} \gg kT$.

3.2.2. Free-bound transitions. The bound state of the two-level atom will be denoted by "1", the ionized state by "+".

We define transition functions $R_{1+}(E; E', E'')$ and $R_{+1}(E', E''; E)$ in the following way: The number of collisional ionizations per unit volume and unit time produced by electrons with energies in the range $(E, \mathrm{d}E)$ such that after the collision the two outgoing electrons have energies in the ranges $(E', \mathrm{d}E')$ and $(E'', \mathrm{d}E'')$, respectively, is given by

$$n_1 n_e (2E/m)^{1/2} f(E)\, \mathrm{d}E\, R_{1+}(E; E', E'')\, \mathrm{d}E'\, \mathrm{d}E'',$$

and the number of corresponding three-body recombinations per unit volume and unit time by

$$n_+ n_e^2 (2E'/m)^{1/2} (2E''/m)^{1/2} f(E')\, f(E'')\, \mathrm{d}E'\, \mathrm{d}E'' \cdot R_{+1}(E', E''; E)\, \mathrm{d}E$$

(n_+ = ion density). The cross section for collisional ionization is hence

$$Q_{1+}(E) = \int\!\!\int R_{1+}(E; E', E'')\, \mathrm{d}E'\, \mathrm{d}E''. \tag{5}$$

The collision term due to ionization and three-body recombination collisions is then for energies $E \geqq$ ionization energy E_{1+} given by

$$\left(\frac{\partial f}{\partial t}\right)_{\text{inel}}^{\text{fb}} = -n_1 \left(\frac{2E}{m}\right)^{1/2} Q_{1+}(E)\, f(E) + n_+ n_e \int\!\!\int \left(\frac{2E'}{m}\,\frac{2E''}{m}\right)^{1/2} \cdot R_{+1}(E', E''; E)\, f(E')\, f(E'')\, \mathrm{d}E'\, \mathrm{d}E'' \tag{6}$$

where again two further terms have been neglected [cf. the discussion of Eq. (4)].

3.3. *Stationarity Equation*

We introduce dimensionless energies η through

$$\eta = E/kT, \quad \eta_{12} = E_{12}/kT, \quad \eta_{1+} = E_{1+}/kT, \tag{7}$$

dimensionless cross sections q through

$$Q_{12}(E) = \pi a_0^2\, q_{12}(\eta), \quad Q_{1+}(E) = \pi a_0^2\, q_{1+}(\eta) \tag{8}$$

($a_0 = \hbar^2/m e_0^2$ = Bohr radius), and the dimensionless transition function r_{1+} through

$$R_{1+}(E; E', E'') = \frac{\pi a_0^2}{(kT)^2}\, r_{1+}(\eta; \eta', \eta''), \tag{9}$$

so that

$$q_{1+}(\eta) = \int\!\!\int r_{1+}(\eta; \eta', \eta'')\, \mathrm{d}\eta'\, \mathrm{d}\eta''. \tag{10}$$

We further define dimensionless numbers α and β by

$$\frac{n_2}{n_1} = \alpha \frac{g_2}{g_1} \exp\{-E_{12}/kT\}, \tag{11}$$

$$\frac{n_e n_+}{n_1} = \beta \frac{2 g_+}{g_1} \left(\frac{m k T}{2\pi\hbar^2}\right)^{\frac{3}{2}} \exp\{-E_{1+}/kT\} \tag{12}$$

(g_1, g_2, g_+ = statistical weigths), and a dimensionless electron distribution function γ by

$$f(E) = \gamma(\eta) \frac{2 E^{1/2}}{\pi^{1/2} (kT)^{3/2}} \exp\{-E/kT\}. \tag{13}$$

Thus $\alpha = 1$ corresponds to a Boltzmann distribution, $\beta = 1$ to a Saha distribution, and $\gamma = 1$ to a Maxwell distribution.

The stationarity Eq. (1) takes in terms of these dimensionless quantities a simple form. Making use of the relations

$$Q_{21}(E) = \frac{g_1}{g_2} \frac{E + E_{12}}{E} Q_{12}(E + E_{12}), \tag{14}$$

$$R_{+1}(E', E''; E) = \frac{g_1}{2 g_+} \frac{\pi^2 \hbar^3}{m} \frac{E}{E' E''} R_{1+}(E; E', E''), \tag{15}$$

which follow in the usual way via the principle of detailed balancing, one finds for the bound-bound case

$$\gamma''(\eta) - \gamma'(\eta) - \frac{n_1/n_e}{2 \ln \Lambda} \left(\frac{a_0}{e_0^2/kT}\right)^2 \eta\, q_{12}(\eta) \cdot [\gamma(\eta) - \alpha\, \gamma(\eta - \eta_{12})] = 0 \tag{16}$$

and for the free-bound case

$$\gamma''(\eta) - \gamma'(\eta) - \frac{n_1/n_e}{2\ln\Lambda}\left(\frac{a_0}{e_0^2/k\,T}\right)^2 \eta\, q_{1+}(\eta) \cdot \left[\gamma(\eta) - \beta\,\frac{1}{q_{1+}(\eta)}\int\!\!\int r_{1+}(\eta;\eta',\eta'')\,\gamma(\eta')\,\gamma(\eta'')\,d\eta'\,d\eta''\right] = 0\,. \tag{17}$$

Eqs. (16) and (17) are valid for $\eta \geqq \eta_{\text{thr}}$ where η_{thr} corresponds to the threshold energy (i. e. $\eta_{\text{thr}} = \eta_{12}$ or η_{1+}). They have to be supplemented by the equation

$$\gamma(\eta) = 1 \quad \text{if} \quad \eta \leqq \eta_{\text{thr}} \tag{18}$$

which merely expresses the fact that below threshold energy the electron distribution is Maxwellian – an assumption already inherent in the elastic collision term (2). In addition, there is the boundary condition that $\gamma(\infty)$ be finite.

Eqs. (16) and (17) show the correct thermodynamic behaviour: The Maxwell distribution $\gamma(\eta) \equiv 1$ obtains only when $\alpha = \beta = 1$, i. e. when the atomic states are in thermal equilibrium, and vice versa. Generally, one has $\alpha, \beta \leqq 1$ and $\gamma(\eta) \leqq 1$ for $\eta \geqq \eta_{\text{thr}}$.

4. Electron Distribution Function

4.1. Differential Equation

The stationarity Eqs. (16) and (17) are integro-differential equations [the kernel of Eq. (16) being a delta function]. No attempt will be made to solve them in that form. However, a lower and an upper bound of the exact solution γ can readily be obtained in terms of the solution γ_0 of an ordinary differential equation. This is a special advantage of the forms (16) and (17) of the stationarity equation.

Putting in the last terms of Eqs. (16) and (17) (i. e. in the terms containing α and β) $\gamma = 0$ and $\gamma = 1$, respectively, one obtains, taking Eq. (10) into account, differential equations whose solutions will be denoted by γ_0 and γ_1, respectively, i. e. one has

$$\gamma_0'' - \gamma_0' - F(\eta)\,\gamma_0 = 0\,, \tag{19}$$

$$\gamma_1'' - \gamma_1' - F(\eta)\,[\gamma_1 - \zeta] = 0 \tag{20}$$

where $\zeta = \alpha$ or β and

$$F(\eta) = \frac{n_1/n_e}{2\ln\Lambda}\left(\frac{a_0}{e_0^2/k\,T}\right)^2 \eta\, q(\eta)\,. \tag{21}$$

The boundary conditions are

$$\gamma_0(\eta_{\text{thr}}) = 1\,, \qquad \gamma_0(\infty) \text{ finite} \tag{22}$$

and identical conditions for $\gamma_1(\eta)$. It follows immediately that γ_0 and γ_1 are related to each other through

$$\gamma_1(\eta) = (1-\zeta)\,\gamma_0(\eta) + \zeta\,. \tag{23}$$

On the other hand, one has the inequality

$$\gamma_0(\eta) \leqq \gamma(\eta) \leqq \gamma_1(\eta)\,. \tag{24}$$

Therefore, a lower and an upper bound of γ can be constructed by means of γ_0. Note that $\gamma = \gamma_1$ in $\eta_{\text{thr}} \leqq \eta \leqq 2\,\eta_{\text{thr}}$ because of Eq. (18). The distribution function γ_0 corresponds to the limiting case of vanishing occupation number of the upper atomic level.

4.2. Cross Sections

4.2.1. Excitation cross section. We write the collisional excitation cross section of an optically allowed transition

$$q_{12}(\eta) = \left(\frac{E_0}{E_{12}}\right)^2 f_{12}\,\varphi\left(\frac{\eta}{\eta_{12}}\right) \tag{25}$$

($E_0 = e_0^2/a_0$ = atomic unit of energy; f_{12} = absorption oscillator strength). The functions φ describing the functional dependence of the cross section on energy are then of the same order of magnitude.

For neutral atoms we choose [8]

$$\varphi_0(u) = \frac{u-1}{u^2}\ln(1{,}25\,u) \qquad (u = \eta/\eta_{\text{thr}}) \tag{26a}$$

which vanishes at threshold $u = 1$ and has its maximum value $\cong 0{,}3$ at $u \cong 4$. For positive ions we choose either

$$\varphi_1(u) = 0{,}4 \tag{26b}$$

or, for some resonant transitions of lithium-like and sodium-like ions [9],

$$\varphi_2(u) = 0{,}4\,K/u\,, \tag{26c}$$

K being a numerical factor in the range 3 ... 10. The functions φ_1 and φ_2 approximate cross sections that are finite at threshold in intervals $1 \leqq u \leqq 4$.

4.2.2. Ionization cross section. We take as collisional ionization cross section of atoms and ions

$$q_{1+}(\eta) = \left(\frac{E_0}{E_{1+}}\right)^2 f_1\,\varphi_0\left(\frac{\eta}{\eta_{1+}}\right) \tag{27}$$

(f_1 = number of equivalent electrons of the energetically highest shell) where φ_0 is given by Eq. (26a).

[8] H. W. Drawin, Z. Phys. **164**, 513 [1961].

[9] B. L. Moiseiwitsch and S. J. Smith, Rev. Mod. Phys. **40**, 238 [1968].

4.3. Electron Distribution Functions

Eq. (21) shows that the combination $\eta\, q(\eta)$ enters the differential Eq. (19). Now, it turns out that for $1 \leqq u \leqq 3$

$$u\, \varphi_0(u) \cong 0{,}45\,(u-1) \tag{28}$$

is a very good approximation. We thus get for $F(\eta)$ defined by Eq. (21)

$$F_0(\eta) = \varkappa(\eta - \eta_{\text{thr}}), \tag{29 a}$$
$$F_1(\eta) = \varkappa\, \eta, \tag{29b}$$
$$F_2(\eta) = K\, \varkappa\, \eta_{\text{thr}}, \tag{29c}$$

corresponding to φ_0, φ_1, and φ_2, respectively. Eqs. (29) are valid approximations for $1 \leqq \eta/\eta_{\text{thr}} \lessapprox 3$, and the quantity $\varkappa$ is given by

$$\varkappa = \frac{0{,}2\,(n_1/n_e)\, f}{\eta_{\text{thr}}^2 \ln \Lambda} \tag{30}$$

where $f = f_{12}$ or f_1.

The corresponding solutions of the differential Eq. (19) with boundary conditions (22) are then

$$\gamma_0^{(0)}(\eta) = \exp[\tfrac{1}{2}(\eta - \eta_{\text{thr}})]\, \frac{\mathrm{Ai}[\varkappa^{1/3}(\eta - \eta_{\text{thr}}) + \frac{1}{4}\varkappa^{-2/3}]}{\mathrm{Ai}[\frac{1}{4}\varkappa^{-2/3}]}, \tag{31a}$$

$$\gamma_0^{(1)}(\eta) = \exp[\tfrac{1}{2}(\eta - \eta_{\text{thr}})]\, \frac{\mathrm{Ai}[\varkappa^{1/3}\eta + \frac{1}{4}\varkappa^{-2/3}]}{\mathrm{Ai}[\varkappa^{1/3}\eta_{\text{thr}} + \frac{1}{4}\varkappa^{-2/3}]}, \tag{31b}$$

$$\gamma_0^{(2)}(\eta) = \exp\{-\tfrac{1}{2}(\eta - \eta_{\text{thr}})\,[(4\,K\,\varkappa\,\eta_{\text{thr}} + 1)^{1/2} - 1]\} \tag{31c}$$

where $\mathrm{Ai}(x)$ is the tabulated Airy function [10]. Examples of distribution functions $\gamma_0^{(0)}$ and $\gamma_0^{(1)}$ are plotted in Figs. 1 to 4.

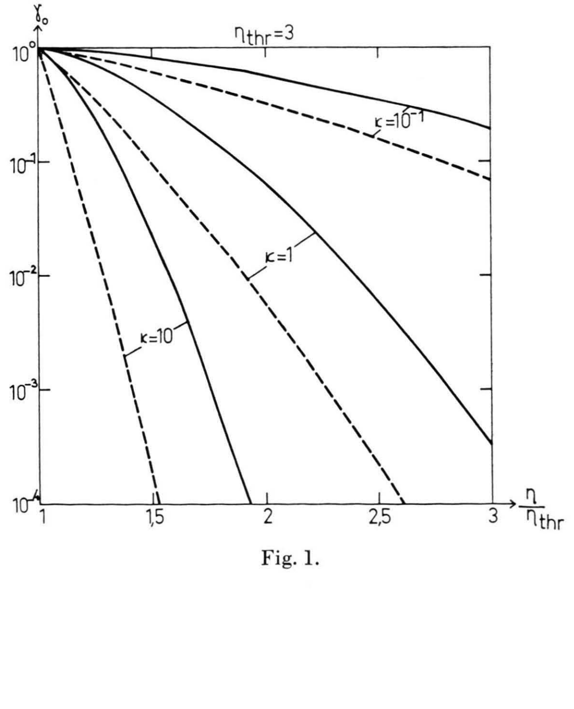

Fig. 1.

5. Collision Rates

The collision rates for excitation and ionization due to an electron distribution function γ_0 will now be compared with those due to a Maxwell distribution. These two collision rates are given by

$$C_0 = \lambda \int_{\eta_{\text{thr}}}^{\infty} \eta\, \varphi(\eta/\eta_{\text{thr}})\, e^{-\eta}\, \gamma_0(\eta)\, \mathrm{d}\eta \tag{32}$$

and

$$C_{\mathrm{M}} = \lambda \int_{\eta_{\text{thr}}}^{\infty} \eta\, \varphi(\eta/\eta_{\text{thr}})\, e^{-\eta}\, \mathrm{d}\eta, \tag{33}$$

respectively, with the same constant λ.

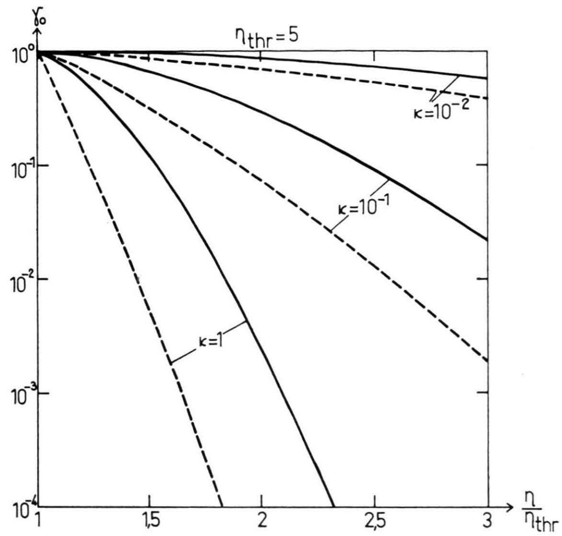

Fig. 2.

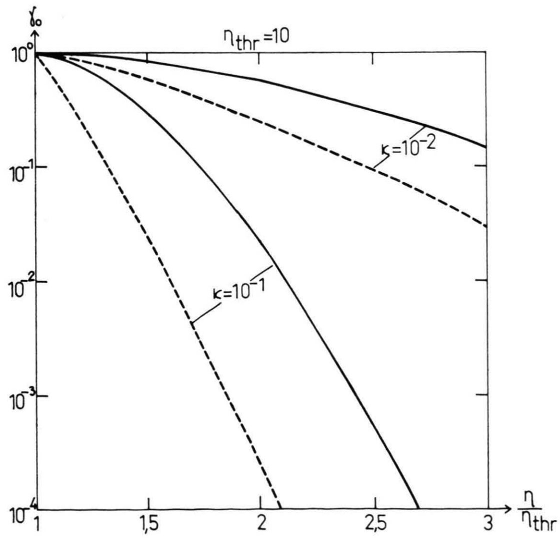

Fig. 3.

[10] M. ABRAMOWITZ and I. A. STEGUN (Editors), Handbook of Mathematical Functions; Dover, New York 1965.

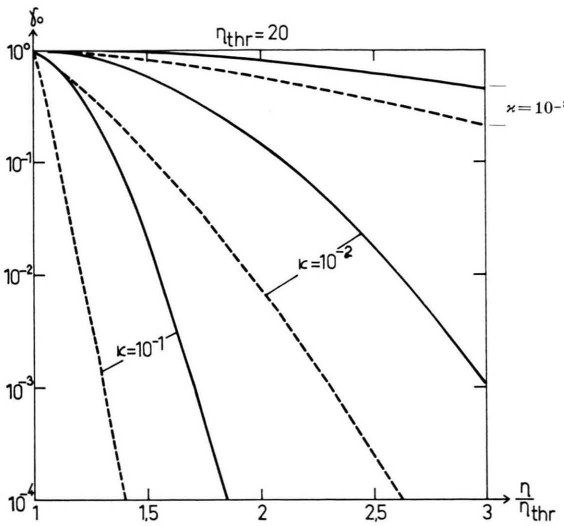

Fig. 4.

Figs. 1 to 4. Reduced electron distribution function γ_0 (=distribution function divided by Maxwell distribution) as a function of energy in threshold units η/η_{thr} for different values of $\eta_{thr}=E_{thr}/k\,T$ and $\varkappa$ [Eq. (30)]. Solid curves correspond to cross section φ_0 [Eq. (26a)] vanishing at threshold, dashed curves to φ_1 [Eq. (26b)] being finite at threshold.

Adopting a procedure due to WOJACZEK[4], we write Eq. (19) in the form

$$\eta\,\varphi(\eta/\eta_{thr})\,\gamma_0 = \frac{0,4}{\varkappa}\,(\gamma_0'' - \gamma_0') \qquad (34)$$

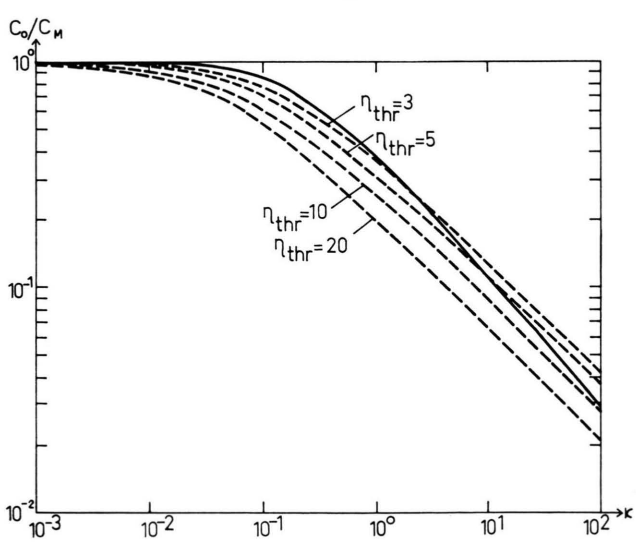

Fig. 5. Ratio C_0/C_M of collision rate due to a non-Maxwellian electron distribution γ_0 to that due to a Maxwell distribution as a function of $\varkappa$ [Eq. (30)] for different values of $\eta_{thr}=E_{thr}/k\,T$. Solid curve (not explicitly dependent on η_{thr}) corresponds to cross section φ_0 [Eq. (26a)] vanishing at threshold, dashed curves to φ_1 [Eq. (26b)] being finite at threshold.

using Eqs. (21), (25) or (27), and (30). It follows that

$$\int_{\eta_{thr}}^{\infty} \eta\,\varphi(\eta/\eta_{thr})\,e^{-\eta}\,\gamma_0(\eta)\,d\eta = -\frac{0,4}{\varkappa}\,\gamma_0'(\eta_{thr})\,e^{-\eta_{thr}}\,. \qquad (35)$$

On the other hand, the corresponding integrals with $\gamma_0=1$ are readily calculated using Eqs. (28), (26b), and (26c). The results are

$$\frac{C_0^{(0)}}{C_M^{(0)}} = -\frac{1}{\varkappa}\left\{\varkappa^{1/3}\,\frac{\mathrm{Ai}'[\frac14\,\varkappa^{-2/3}]}{\mathrm{Ai}[\frac14\,\varkappa^{-2/3}]} + \frac12\right\}, \qquad (36a)$$

$$\frac{C_0^{(1)}}{C_M^{(1)}} = -\frac{1}{\varkappa\,(\eta_{thr}+1)}\left\{\varkappa^{1/3}\,\frac{\mathrm{Ai}'[\varkappa^{1/3}\,\eta_{thr}+\frac14\,\varkappa^{-2/3}]}{\mathrm{Ai}[\varkappa^{1/3}\,\eta_{thr}+\frac14\,\varkappa^{-2/3}]} + \frac12\right\}, \qquad (36b)$$

$$\frac{C_0^{(2)}}{C_M^{(2)}} = \frac{(4\,K\,\varkappa\,\eta_{thr}+1)^{1/2}-1}{2\,K\,\varkappa\,\eta_{thr}}\,, \qquad (36c)$$

where $\mathrm{Ai}'(x)$ is the tabulated derivative of the Airy function[10]. Fig. 5 shows $C_0^{(0)}/C_M^{(0)}$ and some examples of $C_0^{(1)}/C_M^{(1)}$ as functions of $\varkappa$.

6. Discussion

The results obtained lead to following conclusions about deviations of the electron velocity distribution from a Maxwellian one and their effects on collision rates:

1. The deviation increases with increasing parameter $\varkappa \cong 0{,}2\,(n_1/n_e)\,f/\eta_{thr}^2 \ln \Lambda$. Thus only transitions have to be considered whose lower level 1 is the ground state since for excited states the occupation numbers and hence $\varkappa$ are usually much too small to affect the electron distribution.

2. For a given value of $\varkappa$, the deviation increases with increasing η_{thr}, i. e. with decreasing temperature, as inspection of Figs. 1 to 4 shows.

3. The threshold behaviour of the cross sections is of great importance for the effect considered. Quantitative estimates can be taken from Figs. 1 to 4.

4. The collision rate of a given transition is very little affected by the alterations of the electrons distribution that it produces itself. Indeed, Fig. 5 shows that even for as great a value as $\varkappa=1$ the ratio C_0/C_M is still in the range 0,4 ... 0,2 whereas the distribution functions themselves differ tremendously from Maxwellian ones (cf. Figs. 1 to 4). The reason is that for low temperatures ($\eta_{thr} \gtrsim 3$) only the energy range just above threshold energy enters the collision rate. Appreciable changes of collision rates compared to the Maxwellian case may thus be expected only for higher resonance transitions including ionization, and for transitions of other atomic species with high enough threshold energies.